GCE AS MARKING SCHEME

SUMMER 2023

AS
FURTHER MATHEMATICS
UNIT 1 FURTHER PURE MATHEMATICS A
2305U10-1

INTRODUCTION

This marking scheme was used by WJEC for the 2023 examination. It was finalised after detailed discussion at examiners' conferences by all the examiners involved in the assessment. The conference was held shortly after the paper was taken so that reference could be made to the full range of candidates' responses, with photocopied scripts forming the basis of discussion. The aim of the conference was to ensure that the marking scheme was interpreted and applied in the same way by all examiners.

It is hoped that this information will be of assistance to centres but it is recognised at the same time that, without the benefit of participation in the examiners' conference, teachers may have different views on certain matters of detail or interpretation.

WJEC regrets that it cannot enter into any discussion or correspondence about this marking scheme.

WJEC GCE AS FURTHER MATHEMATICS

UNIT 1 FURTHER PURE MATHEMATICS A

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Qu.	Solution	Mark	Notes
1.	Conjugate: $\bar{z} = 3 - \lambda\mathrm{i}$	B1	si
	$(3+\lambda\mathrm{i})^2 + (3-\lambda\mathrm{i})^2 = 2$ $9 + 6\lambda i + i^2\lambda^2 + 9 - 6\lambda i + i^2\lambda^2 = 2$	M1	Attempt to expand
	$9 + 6\lambda\mathrm{i} - \lambda^2 + 9 - 6\lambda\mathrm{i} - \lambda^2 = 2$	A1	
	$2\lambda^2 = 16$ $\lambda = 2\sqrt{2}$ oe (simplified)	A1	eg A0 for $\lambda = \sqrt{\frac{16}{2}}$
		Total [4]	
2. a)	det A = –10	B1	si
	$A^{-1} = \frac{-1}{10}\begin{pmatrix}-7 & 1\\ -4 & 2\end{pmatrix}$	B1	FT their det A
		(2)	
b)	METHOD 1 (Hence): $X = A^{-1}B$	M1	si FT their A^{-1}
	$X = \frac{-1}{10}\begin{pmatrix}-7 & 1\\ -4 & 2\end{pmatrix}\begin{pmatrix}2 & 0 & 9\\ 4 & -20 & 13\end{pmatrix}$ $X = \frac{-1}{10}\begin{pmatrix}-10 & -20 & -50\\ 0 & -40 & -10\end{pmatrix}$	m1	Correct method multiplication
	$X = \begin{pmatrix}1 & 2 & 5\\ 0 & 4 & 1\end{pmatrix}$	A1	cao
	METHOD 2: $\begin{pmatrix}2 & -1\\ 4 & -7\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\end{pmatrix} = \begin{pmatrix}2 & 0 & 9\\ 4 & -20 & 13\end{pmatrix}$	(M1)	Setting up and beginning to multiply matrices
	Leading to $2a - d = 2$, $4a - 7d = 4$ | $2b - e = 0$, $4b - 7e = -20$ | $2c - f = 9$, $4c - 7f = 13$		
	Solving at least 1 set of simultaneous equations, $X = \begin{pmatrix}1 & 2 & 5\\ 0 & 4 & 1\end{pmatrix}$	(m1) (A1)	cao
		(3)	
		Total [5]	

Qu.	Solution	Mark	Notes
3. a)	Another root is $5+\mathrm{i}$	B1 **(1)**	Accept 4 or –4
b)	METHOD 1: $(x-5+\mathrm{i})(x-5-\mathrm{i})$ $x^2-10x+26$ is the quadratic factor $x^4-10x^3+10x^2+160x-416=0$ $(x^2-10x+26)(x^2+ax-16)=0$ $(x^2-10x+26)(x^2-16)=0$ $\therefore x^2-16=0$ Solving, $x=\pm 4$	 M1 A1 m1 A1 A1	 Sum = 10, product = 26
	METHOD 2: Let other two roots be α and β $\therefore 5+\mathrm{i}+5-\mathrm{i}+\alpha+\beta=10$ $\alpha+\beta=0$ $(5-\mathrm{i})(5+\mathrm{i})\alpha\beta=-416$ $26\alpha\beta=-416$ $\alpha\beta=-16$ Solving simultaneous equations, $x=\pm 4$	 (M1) (A1) (A1) (m1) (A1) **(5)** **Total** **[6]**	 1 correct equation 2nd correct equation Or $x^2-16=0$, convincing method If $5-i$ not considered, award SC1 for a use of Factor Theorem SC2 for 1 correct root after FT SC3 for 2 correct roots after FT

Qu.	Solution	Mark	Notes
4. a)	Translation matrix: $\begin{pmatrix}1 & 0 & 2\\0 & 1 & -2\\0 & 0 & 1\end{pmatrix}$ Reflection matrix: $\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}$ $T = \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 2\\0 & 1 & -2\\0 & 0 & 1\end{pmatrix}$ $T = \begin{pmatrix}0 & 1 & -2\\1 & 0 & 2\\0 & 0 & 1\end{pmatrix}$	B1 B1 M1 A1 **(4)**	 FT their translation and reflection matrix cao M0A0 For multiplying the wrong way, which gives $T = \begin{pmatrix}0 & 1 & 2\\1 & 0 & -2\\0 & 0 & 1\end{pmatrix}$
b)	Invariant points given by $\begin{pmatrix}0 & 1 & -2\\1 & 0 & 2\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\1\end{pmatrix} = \begin{pmatrix}x\\y\\1\end{pmatrix}$ Giving, $y - 2 = x$ and $x + 2 = y$. As these are equivalent, there is an infinite number of invariant points.	M1 A1 A1 **(3)** **Total [7]**	FT their T from (a)
5.a)	**AB** = $(-2\mathbf{i} + 7\mathbf{k}) - (3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$ $= -5\mathbf{i} - 4\mathbf{j} + 9\mathbf{k}$ Therefore, $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} + \lambda(-5\mathbf{i} - 4\mathbf{j} + 9\mathbf{k})$ $\mathbf{r} = (3 - 5\lambda)\mathbf{i} + (4 - 4\lambda)\mathbf{j} + (-2 + 9\lambda)\mathbf{k}$	B1 M1 A1 **(3)**	si Accept equivalent convincing
b)	Substituting into plane equation: $2(3 - 5\lambda) + 3(4 - 4\lambda) + 3(-2 + 9\lambda) = 27$ $5\lambda = 15$ $\lambda = 3$ Therefore, point of intersection: $(-12, -8, 25)$	M1 A1 A1 **(3)** **Total [6]**	 FT their λ

Qu.	Solution	Mark	Notes
6.	Putting $z = x + \mathrm{i}y$ $\lvert x + \mathrm{i}y - 3 + \mathrm{i}\rvert = 2\lvert x + \mathrm{i}y - 5 - 2\mathrm{i}\rvert$ $\lvert(x-3) + \mathrm{i}(y+1)\rvert = 2\lvert(x-5) + \mathrm{i}(y-2)\rvert$ $\sqrt{(x-3)^2 + (y+1)^2} = 2\sqrt{(x-5)^2 + (y-2)^2}$ $(x-3)^2 + (y+1)^2 = 4[(x-5)^2 + (y-2)^2]$	M1 m1 A1	
	$x^2 - 6x + 9 + y^2 + 2y + 1$ $= 4x^2 - 40x + 100 + 4y^2 - 16y + 16$	A1	oe
	$3x^2 + 3y^2 - 34x - 18y + 106 = 0$, which is the standard form of a circle.	A1	or equivalent form of a circle $\left(x - \frac{17}{3}\right)^2 + (y-3)^2 = \frac{52}{9}$
	Centre = $\left(\frac{17}{3}, 3\right)$	A1	FT provided coefficients of x^2 and y^2 are equal
		Total [6]	
7.	When $n = 1$, LHS = $\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$ and RHS = $\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$ Therefore, proposition is valid for $n = 1$.	B1	
	Assume result is true for $n = k$ i.e. $\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}^k = \begin{bmatrix} 2^k & 2^{k-1} \times 5k \\ 0 & 2^k \end{bmatrix}$	M1	
	Consider $n = k + 1$ $\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2^k & 2^{k-1} \times 5k \\ 0 & 2^k \end{bmatrix}$	M1 A1	Or $\begin{bmatrix} 2^k & 2^{k-1} \times 5k \\ 0 & 2^k \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$
	$\begin{bmatrix} 2 \times 2^k & (2 \times 2^{k-1} \times 5k) + (5 \times 2^k) \\ 0 & 2 \times 2^k \end{bmatrix}$ Top right entry: $(2^k \times 5k) + (5 \times 2^k)$ $= 2^k(5k + 5)$ $= 2^k \times 5(k + 1)$	A1	
	Therefore, $\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} 2^{k+1} & 2^k \times 5(k+1) \\ 0 & 2^{k+1} \end{bmatrix}$	A1	Remaining 3 entries correct
	If proposition is true for $n = k$, it is also true for $n = k + 1$. As it is true for $n = 1$, by mathematical induction, it is true for all positive integers n.	E1	Award for a perfect solution including the last line.
		Total [7]	

Qu.	Solution	Mark	Notes
8.	$\alpha+\beta+\gamma=-5$ $\alpha\beta+\beta\gamma+\gamma\alpha=2$ $\alpha\beta\gamma=-8$	B1	any two correct equations
	New equation: Sum of roots: $\frac{\alpha}{\beta\gamma}+\frac{\beta}{\gamma\alpha}+\frac{\gamma}{\alpha\beta}$ $=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$ $=\frac{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)}{\alpha\beta\gamma}$ $=\frac{(-5)^2-(2\times 2)}{-8}=-\frac{21}{8}$	M1 A1	Common denominator
	Sum of pairs: $\frac{1}{\gamma^2}+\frac{1}{\beta^2}+\frac{1}{\alpha^2}=\frac{\alpha^2\beta^2+\beta^2\gamma^2+\alpha^2\gamma^2}{\alpha^2\beta^2\gamma^2}$ $=\frac{(\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)}{(\alpha\beta\gamma)^2}$ $=\frac{2^2-(2\times -8\times -5)}{(-8)^2}=\frac{-76}{64}\quad\left(=\frac{-19}{16}\right)$	M1 A1 A1	Common denominator Fully factorised
	Product: $\frac{1}{\alpha\beta\gamma}=-\frac{1}{8}$	B1	
	$\therefore\ \frac{-b}{a}=-\frac{21}{8}$ $\frac{c}{a}=\frac{-76}{64}$ $\frac{-d}{a}=-\frac{1}{8}$	B1	FT previous values, two correct expression
	If $a=1, b=\frac{21}{8}, c=\frac{-76}{64}, d=\frac{1}{8}$ New equation: $x^3+\frac{21}{8}x^2-\frac{76}{64}x+\frac{1}{8}=0$	B1 **Total [9]**	oe FT B1 above eg. If $a=16, b=42, c=-19, d=2$ New equation: $16x^3+42x^2-19x+2=0$

Qu.	Solution	Mark	Notes
9. a)	$u + \mathrm{i}v = 1 - (x + \mathrm{i}y)^2$ $u + \mathrm{i}v = 1 - x^2 + y^2 - 2\mathrm{i}xy$ Comparing coefficients Imaginary parts: $v = -2xy$ Real parts: $u = 1 - x^2 + y^2$	M1 A1 m1 A1 **(4)**	 Both correct
b)	Substituting $y = 4x$ $v = -2x \times 4x = -8x^2$ $u = 1 - x^2 + 16x^2 \quad (= 1 + 15x^2)$ Eliminating x, the equation of the locus Q is $u = 1 + 15\left(\frac{v}{-8}\right)$ oe	M1 A1 M1 A1 **(4)**	FT (a) A1 for both u and v cao $(8u + 15v = 8)$
c)	Point $P(2,5) \to Q(22, -20)$ Equation of the locus of Q is $8u + 15v = 8$ $D = \frac{\lvert(8 \times 22) + (15 \times -20) - 8\rvert}{\sqrt{64 + 225}}$ $D = \frac{132}{17}$ or $7.7647 \ldots$	B1 M1 A1 A1 **(4)** **Total [12]**	FT (a) oe FT their Q (not P) & straight line from (b) cao
10.	METHOD 1: Realisation of difference of two series of cubes $\sum_{r=1}^{k}(2r-1)^3 - \sum_{r=1}^{k-1}(2r)^3$ $= \sum_{r=1}^{k}(8r^3 - 12r^2 + 6r - 1) - \sum_{r=1}^{k-1} 8r^3$ $= \frac{8}{4}k^2(k+1)^2 - \frac{12}{6}k(k+1)(2k+1) + \frac{6}{2}k(k+1) - k - \frac{8}{4}(k-1)^2k^2$ $= k[2k^3 + 4k^2 + 2k - 4k^2 - 6k - 2 + 3k + 3 - 1 - 2k^3 + 4k^2 - 2k]$ $= k^2(4k - 3)$	 B1 M1 A1 A1 m1 A1 A1 A1	 Use of $\sum$ Condone ranges for r other than $r = k$ and $r = k - 1$ for ranges with a difference of 1 Cubing (ignore ranges) Use of sums formulae All correct Simplification Accept $k(4k^2 - 3k)$ or $4k^3 - 3k^2$

Qu.	Solution	Mark	Notes
10.	METHOD 2:		
	$1^3-2^3+3^3-4^3+5^3-6^3+7^3-\cdots$		
	$1^3+2^3+3^3+4^3+5^3+6^3+7^3+\cdots$ $-2(2^3+4^3+6^3+\cdots)$	(B1)	Realisation of difference of sequences
	$1^3+2^3+3^3+4^3+5^3+6^3+7^3+\cdots$ ($2k-1$ terms) $-16(1^3+2^3+3^3+\cdots)$ ($k-1$ terms)	(B1)	Factorising 2^3
	$\sum_{r=1}^{2k-1} r^3 - 16\sum_{r=1}^{k-1} r^3$	(M1) (A1)	Use of $\sum$ Condone ranges for r other than $r=2k-1$ and $r=k-1$ for ranges with a difference of k
	$=\dfrac{(2k-1)^2(2k)^2}{4}-\dfrac{16(k-1)^2k^2}{4}$	(m1) (A1)	Use of sums formulae All correct
	$=k^2[(2k-1)^2-4(k-1)^2]$	(A1)	Simplification
	$=k^2(4k^2-4k+1-4k^2+8k-4)$		
	$=k^2(4k-3)$	(A1)	Accept $k(4k^2-3k)$ or $4k^3-3k^2$
	METHOD 3: Realisation of difference of two series of cubes	 (B1)	
	$\sum_{r=1}^{k}(2r-1)^3-\sum_{r=1}^{k-1}(2r)^3$	(M1) (A1)	Use of $\sum$ Condone ranges for r other than $r=k$ and $r=k-1$ for ranges with a difference of 1
	$=\sum_{r=1}^{k}(8r^3-12r^2+6r-1)-\sum_{r=1}^{k-1}8r^3$	(A1)	Cubing (ignore ranges)
	$=\sum_{r=k}^{k}8r^3+\sum_{r=1}^{k}(-12r^2+6r-1)$		
	$=8k^3-\dfrac{12}{6}k(k+1)(2k+1)+\dfrac{6}{2}k(k+1)-k$	(m1) (A1)	Use of sums formulae All correct
	$=k[8k^2-4k^2-6k-2+3k+3-1]$	(A1)	Simplification
	$=k^2(4k-3)$	(A1)	Accept $k(4k^2-3k)$ or $4k^3-3k^2$
		Total [8]	