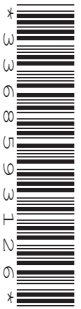


# Advanced Subsidiary GCE (H235)

## Further Mathematics A

### Formulae Booklet



The information in this booklet is for the use of candidates following the Advanced Subsidiary course.

The formulae booklet will be printed for distribution with the examination papers.

Copies of this booklet may be used for teaching.

This document consists of **8** pages.

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## Pure Mathematics

### Binomial series

$$(a+b)^n = a^n + {}^n C_1 a^{n-1} b + {}^n C_2 a^{n-2} b^2 + \dots + {}^n C_r a^{n-r} b^r + \dots + b^n \quad (n \in \mathbb{N}),$$

$$\text{where } {}^n C_r = {}_n C_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

### Matrix transformations

$$\text{Reflection in the line } y = \pm x: \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

$$\text{Anticlockwise rotation through } \theta \text{ about } O: \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Rotations through  $\theta$  about the coordinate axes. The direction of positive rotation is taken to be anticlockwise when looking towards the origin from the positive side of the axis of rotation.

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$\mathbf{R}_y = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$\mathbf{R}_z = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

### Differentiation from first principles

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

### Complex numbers

$$\text{Circles: } |z-a| = k$$

$$\text{Half lines: } \arg(z-a) = \alpha$$

$$\text{Lines: } |z-a| = |z-b|$$

### Vectors and 3-D coordinate geometry

Cartesian equation of the line through the point  $A$  with position vector  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  in direction

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \text{ is } \frac{x-a_1}{u_1} = \frac{y-a_2}{u_2} = \frac{z-a_3}{u_3} (= \lambda)$$

$$\text{Vector product: } \mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & a_1 & b_1 \\ \mathbf{j} & a_2 & b_2 \\ \mathbf{k} & a_3 & b_3 \end{vmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

### Statistics

#### Standard deviation

$$\sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} \text{ or } \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$$

#### Discrete distributions

$X$  is a random variable taking values  $x_i$  in a discrete distribution with  $P(X = x_i) = p_i$

$$\text{Expectation: } \mu = E(X) = \sum x_i p_i$$

$$\text{Variance: } \sigma^2 = \text{Var}(X) = \sum (x_i - \mu)^2 p_i = \sum x_i^2 p_i - \mu^2$$

	$P(X = x)$	$E(X)$	$\text{Var}(X)$
Binomial $B(n, p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	$np$	$np(1-p)$
Uniform distribution over $1, 2, \dots, n$ $U(n)$	$\frac{1}{n}$	$\frac{n+1}{2}$	$\frac{1}{12}(n^2 - 1)$
Geometric distribution $\text{Geo}(p)$	$(1-p)^{x-1} p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson $\text{Po}(\lambda)$	$e^{-\lambda} \frac{\lambda^x}{x!}$	$\lambda$	$\lambda$

#### Non-parametric tests

$$\text{Goodness-of-fit test and contingency tables: } \sum \frac{(O_i - E_i)^2}{E_i} \sim \chi^2_v$$

**Correlation and regression**

For a sample of  $n$  pairs of observations  $(x_i, y_i)$

$$S_{xx} = \sum (x_i - \bar{x})^2 = \sum x_i^2 - \frac{(\sum x_i)^2}{n}, \quad S_{yy} = \sum (y_i - \bar{y})^2 = \sum y_i^2 - \frac{(\sum y_i)^2}{n},$$

$$S_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y}) = \sum x_i y_i - \frac{\sum x_i \sum y_i}{n}$$

Product moment correlation coefficient:  $r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} = \frac{\sum x_i y_i - \frac{\sum x_i \sum y_i}{n}}{\sqrt{\left[\left(\sum x_i^2 - \frac{(\sum x_i)^2}{n}\right)\left(\sum y_i^2 - \frac{(\sum y_i)^2}{n}\right)\right]}}$

The regression coefficient of  $y$  on  $x$  is  $b = \frac{S_{xy}}{S_{xx}} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$

Least squares regression line of  $y$  on  $x$  is  $y = a + bx$  where  $a = \bar{y} - b\bar{x}$

Spearman's rank correlation coefficient:  $r_s = 1 - \frac{6 \sum d_i^2}{n(n^2 - 1)}$

Critical values for Spearman's rank correlation coefficient,  $r_s$ 

$n$	1-Tail Test					2-Tail Test				
	5%	2½%	1%	½%	1%	5%	2½%	1%	½%	
1	-	-	-	-	-	-	-	-	-	-
2	-	-	-	-	-	-	-	-	-	-
3	-	-	-	-	-	-	-	-	-	-
4	1.0000	-	-	-	-	-	-	-	-	-
5	0.9000	1.0000	1.0000	-	-	-	-	-	-	-
6	0.8286	0.8857	0.9429	1.0000	-	-	-	-	-	-
7	0.7143	0.7857	0.8929	0.9286	-	-	-	-	-	-
8	0.6429	0.7381	0.8333	0.8810	-	-	-	-	-	-
9	0.6000	0.7000	0.7833	0.8333	-	-	-	-	-	-
10	0.5636	0.6485	0.7455	0.7939	-	-	-	-	-	-
11	0.5364	0.6182	0.7091	0.7545	-	-	-	-	-	-
12	0.5035	0.5874	0.6783	0.7273	-	-	-	-	-	-
13	0.4835	0.5604	0.6484	0.7033	-	-	-	-	-	-
14	0.4637	0.5385	0.6264	0.6791	-	-	-	-	-	-
15	0.4464	0.5214	0.6036	0.6536	-	-	-	-	-	-
16	0.4294	0.5029	0.5824	0.6353	-	-	-	-	-	-
17	0.4142	0.4877	0.5662	0.6176	-	-	-	-	-	-
18	0.4014	0.4716	0.5501	0.5996	-	-	-	-	-	-
19	0.3912	0.4596	0.5351	0.5842	-	-	-	-	-	-
20	0.3805	0.4466	0.5218	0.5699	-	-	-	-	-	-
21	0.3701	0.4364	0.5091	0.5558	-	-	-	-	-	-
22	0.3608	0.4252	0.4975	0.5438	-	-	-	-	-	-
23	0.3528	0.4160	0.4862	0.5316	-	-	-	-	-	-
24	0.3443	0.4070	0.4757	0.5209	-	-	-	-	-	-
25	0.3369	0.3977	0.4662	0.5108	-	-	-	-	-	-
26	0.3306	0.3901	0.4571	0.5009	-	-	-	-	-	-
27	0.3242	0.3828	0.4487	0.4915	-	-	-	-	-	-
28	0.3180	0.3755	0.4401	0.4828	-	-	-	-	-	-
29	0.3118	0.3685	0.4325	0.4749	-	-	-	-	-	-
30	0.3063	0.3624	0.4251	0.4670	-	-	-	-	-	-

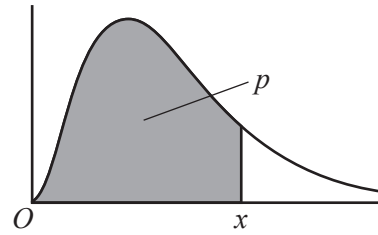
Critical values for the product moment correlation coefficient,  $r$ 

$n$	1-Tail Test					2-Tail Test				
	5%	2½%	1%	½%	1%	5%	2½%	1%	½%	
1	-	-	-	-	-	-	-	-	-	-
2	-	-	-	-	-	-	-	-	-	-
3	0.9877	0.9969	0.9995	0.9999	-	0.3009	0.3550	0.4158	0.4556	0.4487
4	0.9000	0.9500	0.9800	0.9900	-	0.2960	0.3494	0.4093	0.4487	0.4421
5	0.8054	0.8783	0.9343	0.9587	-	0.2913	0.3440	0.4032	0.4421	0.4357
6	0.7293	0.8114	0.8822	0.9172	-	0.2869	0.3388	0.3972	0.4357	0.4296
7	0.6694	0.7545	0.8329	0.8745	-	0.2826	0.3338	0.3916	0.4296	0.4238
8	0.6215	0.7067	0.7887	0.8343	-	0.2785	0.3291	0.3862	0.4238	0.4182
9	0.5822	0.6664	0.7498	0.7977	-	0.2746	0.3246	0.3810	0.4182	0.4128
10	0.5494	0.6319	0.7155	0.7646	-	0.2709	0.3202	0.3760	0.4128	0.4076
11	0.5214	0.6021	0.6851	0.7348	-	0.2673	0.3160	0.3712	0.4076	0.4026
12	0.4973	0.5760	0.6581	0.7079	-	0.2638	0.3120	0.3665	0.4026	0.3978
13	0.4762	0.5529	0.6339	0.6835	-	0.2605	0.3081	0.3621	0.3978	0.3932
14	0.4575	0.5324	0.6120	0.6614	-	0.2573	0.3044	0.3578	0.3932	0.3887
15	0.4409	0.5140	0.5923	0.6411	-	0.2542	0.3008	0.3536	0.3887	0.3843
16	0.4259	0.4973	0.5742	0.6226	-	0.2512	0.2973	0.3496	0.3843	0.3801
17	0.4124	0.4821	0.5577	0.6055	-	0.2483	0.2940	0.3457	0.3801	0.3761
18	0.4000	0.4683	0.5425	0.5897	-	0.2455	0.2907	0.3420	0.3761	0.3721
19	0.3887	0.4555	0.5285	0.5751	-	0.2429	0.2876	0.3384	0.3721	0.3683
20	0.3783	0.4438	0.5155	0.5614	-	0.2403	0.2845	0.3348	0.3683	0.3646
21	0.3687	0.4329	0.5034	0.5487	-	0.2377	0.2816	0.3314	0.3646	0.3610
22	0.3598	0.4227	0.4921	0.5368	-	0.2353	0.2787	0.3281	0.3610	0.3575
23	0.3515	0.4132	0.4815	0.5256	-	0.2329	0.2759	0.3249	0.3575	0.3542
24	0.3438	0.4044	0.4716	0.5151	-	0.2306	0.2732	0.3218	0.3542	0.3509
25	0.3365	0.3961	0.4622	0.5052	-	0.2284	0.2706	0.3188	0.3509	0.3477
26	0.3297	0.3882	0.4534	0.4958	-	0.2262	0.2681	0.3158	0.3477	0.3445
27	0.3233	0.3809	0.4451	0.4869	-	0.2241	0.2656	0.3129	0.3445	0.3415
28	0.3172	0.3739	0.4372	0.4785	-	0.2221	0.2632	0.3102	0.3415	0.3385
29	0.3115	0.3673	0.4297	0.4705	-	0.2201	0.2609	0.3074	0.3385	0.3357
30	0.3061	0.3610	0.4226	0.4629	-	0.2181	0.2586	0.3048	0.3357	0.3328

### Critical values for the $\chi^2$ distribution

If  $X$  has a  $\chi^2$  distribution with  $\nu$  degrees of freedom then, for each pair of values of  $p$  and  $\nu$ , the table gives the value of  $x$  such that

$$P(X \leq x) = p.$$



$p$	0.01	0.025	0.05	0.90	0.95	0.975	0.99	0.995	0.999
$\nu = 1$	0.0 <sup>3</sup> 1571	0.0 <sup>3</sup> 9821	0.0 <sup>2</sup> 3932	2.706	3.841	5.024	6.635	7.879	10.83
2	0.02010	0.05064	0.1026	4.605	5.991	7.378	9.210	10.60	13.82
3	0.1148	0.2158	0.3518	6.251	7.815	9.348	11.34	12.84	16.27
4	0.2971	0.4844	0.7107	7.779	9.488	11.14	13.28	14.86	18.47
5	0.5543	0.8312	1.145	9.236	11.07	12.83	15.09	16.75	20.51
6	0.8721	1.237	1.635	10.64	12.59	14.45	16.81	18.55	22.46
7	1.239	1.690	2.167	12.02	14.07	16.01	18.48	20.28	24.32
8	1.647	2.180	2.733	13.36	15.51	17.53	20.09	21.95	26.12
9	2.088	2.700	3.325	14.68	16.92	19.02	21.67	23.59	27.88
10	2.558	3.247	3.940	15.99	18.31	20.48	23.21	25.19	29.59
11	3.053	3.816	4.575	17.28	19.68	21.92	24.73	26.76	31.26
12	3.571	4.404	5.226	18.55	21.03	23.34	26.22	28.30	32.91
13	4.107	5.009	5.892	19.81	22.36	24.74	27.69	29.82	34.53
14	4.660	5.629	6.571	21.06	23.68	26.12	29.14	31.32	36.12
15	5.229	6.262	7.261	22.31	25.00	27.49	30.58	32.80	37.70
16	5.812	6.908	7.962	23.54	26.30	28.85	32.00	34.27	39.25
17	6.408	7.564	8.672	24.77	27.59	30.19	33.41	35.72	40.79
18	7.015	8.231	9.390	25.99	28.87	31.53	34.81	37.16	42.31
19	7.633	8.907	10.12	27.20	30.14	32.85	36.19	38.58	43.82
20	8.260	9.591	10.85	28.41	31.41	34.17	37.57	40.00	45.31
21	8.897	10.28	11.59	29.62	32.67	35.48	38.93	41.40	46.80
22	9.542	10.98	12.34	30.81	33.92	36.78	40.29	42.80	48.27
23	10.20	11.69	13.09	32.01	35.17	38.08	41.64	44.18	49.73
24	10.86	12.40	13.85	33.20	36.42	39.36	42.98	45.56	51.18
25	11.52	13.12	14.61	34.38	37.65	40.65	44.31	46.93	52.62
30	14.95	16.79	18.49	40.26	43.77	46.98	50.89	53.67	59.70
40	22.16	24.43	26.51	51.81	55.76	59.34	63.69	66.77	73.40
50	29.71	32.36	34.76	63.17	67.50	71.42	76.15	79.49	86.66
60	37.48	40.48	43.19	74.40	79.08	83.30	88.38	91.95	99.61
70	45.44	48.76	51.74	85.53	90.53	95.02	100.4	104.2	112.3
80	53.54	57.15	60.39	96.58	101.9	106.6	112.3	116.3	124.8
90	61.75	65.65	69.13	107.6	113.1	118.1	124.1	128.3	137.2
100	70.06	74.22	77.93	118.5	124.3	129.6	135.8	140.2	149.4

## Mechanics

### Kinematics

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$s = \frac{1}{2}(u + v)t$$

$$v^2 = u^2 + 2as$$

$$s = vt - \frac{1}{2}at^2$$

### Newton's experimental law

Between two smooth spheres  $v_1 - v_2 = -e(u_1 - u_2)$

Between a smooth sphere with a fixed plane surface  $v = -eu$

### Motion in a circle

Tangential velocity is  $v = r\dot{\theta}$

Radial acceleration is  $\frac{v^2}{r}$  or  $r\dot{\theta}^2$  towards the centre

Tangential acceleration is  $\dot{v} = r\ddot{\theta}$

## Discrete Mathematics

### Sorting algorithms

Bubble sort:

Start at the left hand end of the list unless specified otherwise.

Compare the first and second values and swap if necessary. Then compare the (new) second value with the third value and swap if necessary. Continue in this way until all values have been considered.

Fix the last value then repeat with the reduced list until either there is a pass in which no swaps occur or the list is reduced to length 1, then stop.

Shuttle sort:

Start at the left hand end of the list unless specified otherwise.

Compare the second value with the first and swap if necessary, this completes the first pass. Next compare the third value with the second and swap if necessary, if a swap happened shuttle back to compare the (new) second with the first as in the first pass, this completes the second pass.

Next compare the fourth value with the third and swap if necessary, if a swap happened shuttle back to compare the (new) third value with the second as in the second pass (so if a swap happens shuttle back again). Continue in this way for  $n - 1$  passes, where  $n$  is the length of the list.

## Network algorithms

### Dijkstra's algorithm

START with a graph  $G$ . At each vertex draw a box, the lower area for temporary labels, the upper left hand area for the order of becoming permanent and the upper right hand area for the permanent label.

STEP 1 Make the given start vertex permanent by giving it permanent label 0 and order label 1.

STEP 2 For each vertex that is not permanent and is connected by an arc to the vertex that has just been made permanent (with permanent label =  $P$ ), add the arc weight to  $P$ . If this is smaller than the best temporary label at the vertex, write this value as the new best temporary label.

STEP 3 Choose the vertex that is not yet permanent which has the smallest best temporary label. If there is more than one such vertex, choose any one of them. Make this vertex permanent and assign it the next order label.

STEP 4 If every vertex is now permanent, or if the target vertex is permanent, use 'trace back' to find the routes or route, then STOP; otherwise return to STEP 2.

### Prim's algorithm (graphical version)

START with an arbitrary vertex of  $G$ .

STEP 1 Add an edge of minimum weight joining a vertex already included to a vertex not already included.

STEP 2 If a spanning tree is obtained STOP; otherwise return to STEP 1.

### Prim's algorithm (tabular version)

START with a table (or matrix) of weights for a connected weighted graph.

STEP 1 Cross through the entries in an arbitrary row, and mark the corresponding column.

STEP 2 Choose a minimum entry from the uncircled entries in the marked column(s).

STEP 3 If no such entry exists STOP; otherwise go to STEP 4.

STEP 4 Circle the weight  $w_{ij}$  found in STEP 2; mark column  $i$ ; cross through row  $i$ .

STEP 5 Return to STEP 2.

### Kruskal's algorithm

START with all the vertices of  $G$ , but no edges; list the edges in increasing order of weight.

STEP 1 Add an edge of  $G$  of minimum weight in such a way that no cycles are created.

STEP 2 If a spanning tree is obtained STOP; otherwise return to STEP 1.

## Additional Pure Mathematics

### Vector product

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}}$ , where  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\hat{\mathbf{n}}$ , in that order, form a right-handed triple.