Oxford Cambridge and RSA

Monday 27 June 2022 – Afternoon

A Level Further Mathematics B (MEI)

Y436/01 Further Pure with Technology

Time allowed: 1 hour 45 minutes

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a computer with appropriate software
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

Answer **all** the questions.

1 **(a)** A family of curves is given by the equation

$$x^2 + y^2 + 2axy = 1 \ (*)$$

where the parameter a is a real number.
You may find it helpful to use a slider (for a) to investigate this family of curves.

(i) On the axes in the Printed Answer Booklet, sketch the curve in each of the cases

- $a = 0$
- $a = 0.5$
- $a = 2$

[3]

(ii) State a feature of the curve for the cases $a = 0$, $a = 0.5$ that is **not** a feature of the curve in the case $a = 2$. **[1]**

(iii) In the case $a = 1$, the curve consists of two straight lines. Determine the equations of these lines. **[2]**

(b) **(i)** Find an equation of the curve (*) in polar form. **[3]**

(ii) Hence, or otherwise, find, in exact form, the area bounded by the curve, the positive part of the x-axis and the positive part of the y-axis, in the case $a = 2$. **[2]**

(c) In this part of the question m is any real number.

Describing all possible cases, determine the pairs of values a and m for which the curve with equation (*) intersects the straight line given by $y = mx$. **[9]**

2 **(a)** In this part of the question n is an integer greater than 1.

An integer q, where $0 < q < n$ is said to be a quadratic residue modulo n if there exists an integer x such that $x^2 \equiv q \pmod{n}$.

Note that under this definition 0 is not considered to be a quadratic residue modulo n.

(i) Find all the integers x, where $0 \leqslant x < 1000$ which satisfy $x^2 \equiv 481 \pmod{1000}$. **[1]**

(ii) Explain why 481 is a quadratic residue modulo 1000. **[1]**

(iii) Determine the quadratic residues modulo 11. **[2]**

(iv) Determine the quadratic residues modulo 13. **[2]**

(v) Show that, for any integer m, $m^2 \equiv (n-m)^2 \pmod{n}$. **[2]**

(vi) Prove that if p is prime, where $p > 2$, then the number of quadratic residues modulo p is $\frac{p-1}{2}$. **[4]**

(b) Fermat's little theorem states that if p is prime and a is an integer which is co-prime to p, then $a^{p-1} \equiv 1 \pmod{p}$.

(i) Use Fermat's little theorem to show that 91 is not prime. **[2]**

(ii) Create a program to find an integer n between 500 and 600 which is not prime and such that $a^{n-1} \equiv 1 \pmod{n}$ for all integers a which are co-prime to n.
In the Printed Answer Booklet

- Write down your program in full.
- State the integer found by your program.

[6]

3 In this question you are required to consider the family of differential equations

$$\frac{dy}{dx} = \frac{y^a}{x+1} - \frac{1}{y} \quad (*)$$

and its solutions. The parameter a is a real number.

You should assume that $x \geqslant 0$ and $y > 0$ throughout this question.

(a) In this part of the question $a = 1$.

(i) On the axes in the Printed Answer Booklet

- Sketch the isocline defined by $\frac{dy}{dx} = 0$.
- Shade and label the region in which $\frac{dy}{dx} > 0$.
- Shade and label the region in which $\frac{dy}{dx} < 0$.

[3]

(ii) For $b > 0$, find, in terms of b, the solution to (*) which passes through the point $(0, b)$. **[1]**

(iii) Determine

- The values of $b > 0$ for which the solution in **(ii)** has a turning point.
- The corresponding maximum value of y.

[4]

(b) **Fig. 3.1** and **Fig. 3.2** show tangent fields for two distinct but unspecified values of a. In each case a sketch of the solution curve $y = \mathrm{g}(x)$ which passes through $(0, 2)$ is shown for $0 \leqslant x \leqslant 0.5$.

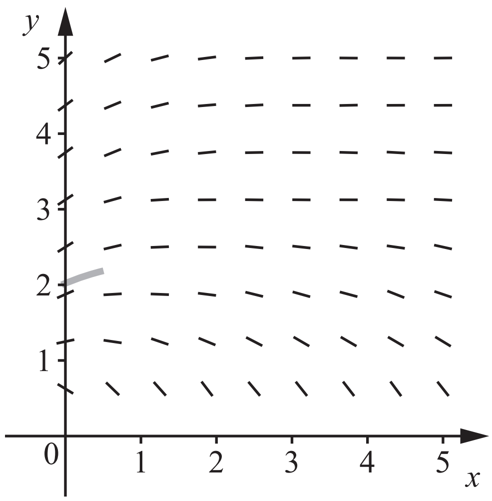

Fig. 3.1

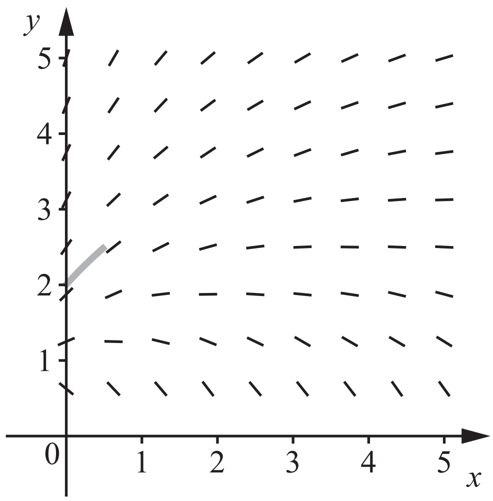

Fig. 3.2

(i) For the case in **Fig. 3.1** suggest a possible value of a. **[1]**

(ii) For the case in **Fig. 3.2** suggest a possible value of a. **[1]**

(iii) In each case, continue the sketch of the solution curves for $0.5 \leqslant x \leqslant 5$ in the Printed Answer Booklet. **[2]**

(iv) State a feature which is present in one of the curves in part **(iii)** for $0.5 \leqslant x \leqslant 5$ but not in the other. **[1]**

(c) **(i)** The Euler method for the solution of the differential equation $\frac{dy}{dx} = f(x, y)$ is as follows

$$y_{n+1} = y_n + hf(x_n, y_n).$$

It is given that $x_0 = 0$ and $y_0 = 2$.

- Construct a spreadsheet to solve (*) using the Euler method so that the value of a and the value of h can be varied, in the case $x_0 = 0$ and $y_0 = 2$.
- State the formulae you have used in your spreadsheet.

[3]

(ii) In this part of the question $a = 0.1$.

Use your spreadsheet with $h = 0.1$ to approximate the value of y when $x = 3$ for the solution to (*) in which $y = 2$ when $x = 0$. **[1]**

(iii) In this part of the question $a = -0.2$.

Use your spreadsheet to approximate, to **1** decimal place, the x-coordinate of the local maximum for the solution to (*) in which $y = 2$ when $x = 0$. **[3]**

END OF QUESTION PAPER

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